Combination Tones as Harmonic Material

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Combination tones, also called sum/difference tones, heterodynes, or Tartini tones are psychoacoustic phenomena created by the interaction of two sounds of sufficient loudness. During the last half century, composers have been working with combination tones as harmonic material. This document investigates Ezra Sims’ *Quintet* and my own composition, *Angelswort*, exploring a few methods of deriving a harmonic language from combination tones. Though most of the works that employ combination tones as harmony are within the realm of electroacoustic music, the focus of this document is on these examples of instrumental music. Methods of deriving harmony, and in some cases melody, from combination tones are analyzed in these two works. Other topics include tuning systems, contrapuntal motion, and consonance/dissonance as they relate to combination tones.
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Introduction: The Acoustics of Combination Tones

Preliminary Terminology

Musicians have been aware of the phenomenon of combination tones for centuries. Eighteenth-century composer and theorist Giuseppe Tartini is often credited with the discovery of the phenomenon. While playing a double stop on his violin, Tartini noticed a terzo suono, a third sound. He was hearing a combination tone, a resultant sound of the two fingered pitches on his instrument, in this case equal to the difference of the two frequencies. Furthermore, he noticed that the third sound could aid him in tuning. By adjusting the two pitches on his instrument to create a consonance with the third tone, he found the optimal tuning of the fingered pitches. Thus, the term “Tartini tone” refers to a pitch heard below two sounding pitches, used as an aid in tuning. The term “generating tones” will be used hereafter to refer to the initial sounding pitches, in this case, the fingered pitches of the double stop.

Musicians are also familiar with the term “beating”, which refers to regular fluctuations in the volume of two sounds whose frequencies of oscillation are similar. As two sounds diverge from a perfect unison we hear an increase in beating. Performers consciously or unconsciously adjust to eliminate beats as a method of tuning.

Arthur Benade’s seminal work Fundamentals of Musical Acoustics uses the term “heterodynes” to refer to any sounds perceived as a result of two other sounds¹. In this document, the general term “combination tones” will be used to refer to all of the above.

phenomena as well as any other resultant frequencies from the combination of two 
sounds. The terms “sum tone” and “difference tone” refer to distinct identities within the 
larger category of combination tones.

The Production of Combination Tones

To demonstrate the phenomenon of combination tones, let us imagine that we are 
holding two tuning forks near one of our ears. One of the forks is tuned to A 440 Hz, 
meaning it oscillates 440 times per second when struck. The second fork, perhaps a 
factory reject, is tuned to 441 Hz (or 441 cycles-per-second). In musical terms, this 
disparity is about four cents, meaning 4% of one semitone. When both forks are struck 
simultaneously and held up to the ear, the sound is perceived as a single pitch.
Humans can recognize a difference between two pitches that are three to six cents 
apart², depending on the individual. Since they are sounding simultaneously, the aural 
effect is a unison.

The unison will beat (fluctuate in volume) at a predictable frequency equal to the 
difference of the two generating frequencies, 1 Hz. Since the lower threshold of human 
pitch perception is around 20 Hz, a sound oscillating at 1 Hz is not perceived as a pitch 
(as in the Tartini Tone) but as a slow rhythm, like a metronome pulsing at 60 bpm, the 
tempo equivalent of 1 Hz. This slow fluctuation can still be described as a “difference 
tone” even though it is perceived as a rhythmic pulsing rather than a distinct pitch. The 
process is the same. This experiment also demonstrates that the difference tone is not

actually a separate sound, but rather, a fluctuation in the loudness of the composite unison created by the two forks. They are beating at a rate of one beat-per-second.

In order to fully explain the phenomenon of this difference tone, we must first understand the physics of sound waves and phase cancellation. One can observe a sine wave on an oscilloscope tracing a path from its highest amplitude (peak, or +1), to its lowest amplitude (trough, or -1) crossing zero in between. The path from zero to +1, to zero, to -1, and back to zero completes one cycle of the sine wave. For our 1 Hz difference tone, that process takes precisely one second. The amplitude of the sound is perceived as the absolute value of the signal. Therefore, both the peaks and the troughs are perceived as full volume, while the zero-crossings are perceived as silence.

When two sounds are simultaneously perceived, the loudness of their respective sound waves are summed together. In our tuning fork example, when the two forks are struck, the oscillations begin “in-phase”, meaning that the peaks and troughs are aligned. At this moment, the composite sound is perceived as louder than the volume of the single fork (Figure 1). The result of the summation is the absolute value of 1 + 1 and -1 + -1 at the peaks and troughs, which in both cases is two. Both signals cross the 0-axis at the same time. This is known as constructive interference. If our forks were both tuned to 440 Hz, the oscillation would continue this way. As the mistuned forks continue to oscillate, they become increasingly “out-of-phase” until after one half-second, the peak of the 441 Hz fork aligns with the trough of the 440 Hz fork. At this moment, the summation of the two signals equals the absolute value of 1 + -1 and -1 + 1, which in both cases equals zero (Figure 1). This phenomenon is called destructive
interference or phase cancellation. Since the two oscillators cancel each other out, the net result is silence.

![Figure 1. Composite Loudness of Two Sound Waves, Phase Cancellation](image)

The two moments described above occur for only about a millisecond. The two mistuned forks begin in-phase, gradually move out-of-phase at 0.5 seconds, then gradually move back into phase in order to start the cycle again at the end of one second. Therefore, the amplitude of the composite sound scales from two to zero and back to two every second. We perceive the composite sound as beating, or pulsing at the frequency of the difference between the two input signals.

Imagine we performed the same experiment again using our standard 440 Hz tuning fork but replaced the 441 Hz fork with a fork tuned to 495 Hz, approximately an
equal-tempered B. If we struck the two forks simultaneously we would perceive the interval of the major second instead of a composite unison. We also notice that the beating disappears and is replaced by another phenomenon, the so-called Tartini tone. Again, we would hear a frequency equal to the difference of the two sounds. In this case, the difference tone is A 55 Hz, three octaves below A 440 Hz. The two pitches are far enough apart that their interference is perceived as a pitch rather than a rhythm.

These two tuning fork examples produce two different perceptual results through the same physical process. In the second example we still perceive fluctuations of loudness in the composite major second, but they are rapid enough to be perceived as a separate pitch. The difference tone, the specific combination tone perceived by Tartini and used in our tuning fork example is often the most readily audible of all combination tones. However, there are an infinite number of combination tones that could be perceived depending on the acoustical circumstances, though many of them will be extremely quiet.

The Importance of Tuning Systems

All individual acoustic\(^3\) sounds are actually a composite of many constituent frequencies called partials. The lowest (and usually loudest) partial in a sound’s spectrum is referred to as its fundamental, which we perceive as the sole pitch. As we move up the frequency spectrum, the partials become quieter. In most sounds\(^4\), the

\(^3\) As opposed to synthesized sine waves.

\(^4\) Those with partials that are not whole number multiples of a single fundamental are referred to as “inharmonic”.
partials are “harmonics” (or overtones) that are whole number multiples of the fundamental pitch\(^5\).

Except for the octave, there is no interval shared between the harmonic series and our 12-Tone Equal Temperament tuning system. This discrepancy has lead some composers to use tuning systems based on harmonic ratios, also called Just Intonation. This term refers to any tuning scheme derived from harmonics, not a single scale or tuning system. The rudimentary pitches in Just Intonation are harmonics labeled using whole numbers. Intervals and chords are described using whole number ratios, such as the major triad 6:5:4 and the dominant-7th chord 7:6:5:4.

With Just intervals, combination tones reinforce the larger tonality. Since all harmonics are multiples of a given fundamental, all combination tones created from the interaction of harmonics will result in other harmonics above the same fundamental. In the second tuning fork example above, the two forks, 440 Hz and 495 Hz have a frequency ratio of 9:8, also called the Just “major whole tone”\(^6\), so the difference tone of 55 Hz is exactly equivalent to the fundamental. Since harmonic ratios are equivalent to frequency ratios, 495-440=55 can be simplified in Just Intonation as 9-8=1.

Equal-tempered intervals, on the other hand, create combination tones that often bear no resemblance to the generating tones and are not themselves equal-tempered. Equal-tempered intervals vary in their deviation from harmonic intervals. Certain equal-tempered intervals, such as the perfect 5th, are fair approximations of harmonic intervals, and produce combination tones that reinforce a kind of quasi-harmonic


system. The equal-tempered P5 is only two cents lower than the Just 3:2, which is a
virtually imperceptible difference to the ear. Other equal-tempered intervals, such as
the major third, are more noticeably inharmonic. The equal-tempered major third is 14
cents above the Just 5:4. In Figure 2 below, these two intervals are shown, first in Just
Intonation and then in Equal Temperament. The sum and difference tones are shown
above and below each pair of input pitches.

![Figure 2. Just Vs. Equal-Tempered P5 and M3 with Combination Tones.](image)

The two-cent deviation in the perfect fifth generating interval creates a negligible
one-cent deviation in the sum tone and a barely noticeable six-cent deviation in the
difference tone. The 14-cent deviation in the major third generator creates a smaller
ten-cent offset in the sum and an enormous 67-cent deviation in the difference.
The discrepancy between the relative tuning of the sum tone and difference tone in Figure 2 is caused by our hearing mechanism. We perceive frequency logarithmically, meaning that each doubling in frequency is perceived as a linear change in pitch of one octave. Therefore, an identical variation in frequency equates a larger pitch offset in low registers than in high.

**Combination Tone Orders**

In discussing combination tones it will be useful to develop a nomenclature that points to their origin. Hereafter, \([P]\) will refer to the lower-pitched and \([Q]\) will refer to the higher-pitched of the two generating tones. These are the variables used by Arthur Benade in *Fundamentals of Musical Acoustics*. These letters are somewhat arbitrary, but they avoid conflict with note names and common mathematical variables.

In the process of calculating combination tones one may encounter a negative answer for a difference equation. The case is similar to the amplitude of a sine wave oscillating between -1 and 1. Only the absolute value is relevant. In other words, if \([P-Q]\) is negative, we can substitute \([Q-P]\), which is positive. Since we are only considering absolute value, the four possible equations, \([P+Q]\), \([Q+P]\), \([P-Q]\), and \([Q-P]\), result in only two answers.

When discussing combination tones, I will refer to “orders” of component frequencies, as suggested by Benade in *Fundamentals of Musical Acoustics*. The first

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7 The brackets are used to enclose a single perceived pitch, the contents of the brackets illustrate its origin in relation to the generating tones, \(P\) and \(Q\).

order is comprised of the two original fundamentals, \([P]\) and \([Q]\). The second order contains the combinations of those two pitches, \([P+Q]\) (also called the “sum tone”) and \([P-Q]\) (also called the “difference tone”).

As previously discussed, in an acoustic system the harmonics of both fundamentals will always be present as well. Therefore, \([2P]\), \([2Q]\), \([3P]\), \([3Q]\), \([4P]\), \([4Q]\), etc. are assumed. The third order involves the interactions of one fundamental with a second harmonic, \([2P+Q]\), \([2Q+P]\), \([2Q-P]\), and \([2P-Q]\). Keep in mind that \([Q-2P]\) and \([P-2Q]\) are identical to \([2P-Q]\) and \([2Q-P]\) respectively, because of the absolute value rule described above. These same combination tones could also be described as the interaction of one combination tone from the second order and one generating tone. For example, if \([P+Q]\) is combined with \([Q]\), the resulting combination tones would be \([2Q\pm P]\). Third-order combination tones are less discussed, because they are less often audible.

The fourth order consists of \([2P+2Q]\) (or two times the sum tone), \([2P-2Q]\) (or two times the difference), \([3P\pm Q]\), and \([3Q\pm P]\). These are derived using two elements from the second order, or one element from the third order and one element from the first order. The fifth order contains \([2P\pm 3Q]\), \([2Q\pm 3P]\), \([P\pm 4Q]\), and \([Q\pm 4P]\). This process continues through an infinite number of orders.

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9 This particular combination tone \([2P-Q]\), has earned its own name, the “cubic difference tone”, because it is often easily perceptible.
Benade explains the first three orders, and states that higher orders are “musically negligible”\(^\text{10}\). Indeed, most musical examples do not reach beyond the third combination tone order, but in certain acoustic and compositional scenarios, which will be discussed in the following analyses, higher orders are absolutely relevant.

Because they involve harmonics that are proportionally quieter than the fundamental, the combination tones from the third order and higher are quieter than the lower orders. Still, upper-order pitches are audible in certain circumstances.

A naturally occurring example of upper-order combination tones is the complex timbre of woodwind multiphonics. Multiphonics are a result of the instrument attempting to oscillate at two frequencies simultaneously\(^\text{11}\). The clash of the two sounds result in mathematically predictable combination tones. Similarly, when one whistles while singing, or sings through an instrument while producing a normal tone simultaneously, the same phenomenon occurs.

A recording of harmonic number 244 from Peter Veale and Claus-Steffen Mahnkopf’s _The Techniques of Oboe Playing_ was analyzed for pitch in Max/MSP using the \(~\text{fiddle}\) object. This analysis yielded 29 specific pitches audible in the multiphonic. Perceptually, the sound is a clangorous ringing, similar to that of a bell. Present in that gestalt sound are the individual combination tones seen in Figure 3. I assigned \(P\) and \(Q\) to the two loudest component pitches: \(P\) is 509 Hz and \(Q\) is 709 Hz.

\(^{10}\) Ibid., 256-257.

\(^{11}\) Peter Veale and Claus-Steffen Mahnkopf, _The Techniques of Oboe Playing_, 69-71.
Figure 1. Oboe Multiphonic Spectral Analysis

Frequencies: 509 709 909 1018 1218 1527 1727 2036 2236 2436 2746 2946 3146 3654 3963 4165 4471 4674 4872 5383 5895 6406

Analysis: [P] [Q] [2Q-P] [2P] [P+Q] [3P] [2P+Q] [2Q+P] [4P] [3P+Q] [3P+2Q] [4P+Q] [3Q+2P] [3Q+3P] [5P+2Q] [4P+3Q] [3P+2Q] [5P+3Q] [4Q+4P] [5P+4Q] [6P+4Q] [7P+4Q]

Figure 3. Oboe Multiphonic Spectral Analysis with microtonal symbols from Hjertmann’s *Angelswort*
Notice the prevalence of sum tones in this example, and the absence of the difference tone. The formants created by the physical shape and construction of the oboe account for the particular collection of combination tones that are present\(^{12}\). Because each instrument has unique areas of the frequency spectrum that are prominent and others that are attenuated, the combination tones that fall within those frequency bands are strengthened or attenuated respectively.

Given the presence of 3654 Hz \([3P+3Q]\) and 4872 Hz \([4P+4Q]\), we can recognize the presence of whole number multiples of the sum tone, which could also be labeled as \([3(P+Q)]\) and \([4(P+Q)]\). The same can occur with harmonics of the difference tone. We also notice a linearity to the combination tones, many of which are exactly 200 Hz apart, the frequency of the difference tone. This provides yet another way to examine the combination tone phenomenon. The harmonic series of the difference tone (200 Hz) could also be described as a scale in which each step is 200 Hz apart. Most of the multiphonic’s constituent pitches can be described as combinations of the difference tone and \([P]\). The first eleven pitches in the figure above could be called \([P]\), \([P+\text{Diff}]\), \([P+2(\text{Diff})]\), \([2P]\), \([2P+\text{Diff}]\), \([3P]\), \([3P+\text{Diff}]\), \([3P+2(\text{Diff})]\), \([4P]\), \([4P+\text{Diff}]\), \([4P+2(\text{Diff})]\).

Any given combination tone in a system, e.g., \([3P-2Q]\), can be theoretically derived from the interaction of the harmonics of \(P\) and \(Q\), the interaction of another combination tone and a fundamental or harmonic, e.g., \([(2P-2Q)+P]\), or the interaction between multiple combination tones, e.g., \([(P-Q)+(2P-Q)]\). Because all the heterodynes are arithmetically related, one could assign the names \(P\) and \(Q\) to other pitches and still

\(^{12}\) ibid.
derive all the remaining pitches. In Figure 3 for example, one could assign [P] to 709 Hz and [Q] to 1218 Hz. The pitches could then be described in order from lowest-to-highest as: [P-Q], [P], [3P-Q], [2Q-2P], [Q], [3Q-3P], [2Q-P], [P+Q], [4Q-4P], [3Q-2P], [2Q], [4Q-3P], [3Q-P], [2Q+P], [3Q], [5Q-3P], [4Q-P], [6Q-4P], [5Q-2P], [4Q], [5Q-P], [6Q-2P], and [7Q-3P]. The initial analysis seen in Figure 3 is preferable only because it involves lower harmonics and uses adjacent tones as the generators.

The generating interval in the multiphonic, 709:509, is about 9 cents smaller than a Just “septimal tritone”\textsuperscript{13}, 7:5 of a fundamental around 101 Hz. As in Figure 2, the ear recognizes the close approximation and the listener may perceive a fundamental around 101 Hz or second harmonic around 202 Hz. Some combination tone systems, like this one, so closely approximate harmonic systems that they will be perceived as stretched or colored harmonic series. In this case, we hear a relationship that suggests a fundamental below the normal range of the oboe.

**Compositional Applications of Combination Tones**

The earliest compositional applications of combination tones date to the 1950's with the use of amplitude modulation, also called ring modulation. Ring modulation is an effect which modulates the amplitude of an input signal by the amplitude of the carrier signal. This is the same type of interference which occurred in the tuning fork example, generating sum and difference tones. These mixed with the input signal, creating a harmonization of the input. Karlheinz Stockhausen was one of

the first composers to use this effect as an element in composition. The earliest examples include *Mixtur* (1964), in which he applied live ring modulation to orchestral groups using microphones and sine tone generators\(^{14}\), and *Mikrophonie II* (1965) in which he combined the signals of a live Hammond organ with a choir of voices.

The bulk of compositional practice involving combination tones followed from the use of ring modulation within the electroacoustic realm. However, the focus of this document is two works for acoustic instruments that employ combination tones not as an effect, but as a method of controlling harmony and counterpoint. Early experiments with this approach include Gérard Grisey’s *Partiels* (1975) that uses combination-tone harmony as a means of transition from noise sounds to pitched sounds\(^{15}\). Tristan Murail’s *Ethers* (1978) uses combination tones as a harmonic device controlling the pitch of a solo flute and ensemble. Claude Vivier used a similar technique to create what he called “les couleurs” in *Lonely Child* (1980)\(^{16}\) and many of his late works.

Maryanne Amacher used the microsonic inner-ear emissions (otoacoustics) of her audience, including them as direct participants in the combination-tone harmony. La Monte Young and Horațiu Rădulescu have capitalized on the shamanistic element of combination tones as a musico-mystical entry point for composition. La Monte Young refers to combination tones in connection with the Yogic concept of *anahata nada*\(^{17}\), the

\(^{14}\) Karlheinz Stockhausen, "Electroacoustic Performance Practice", 74-105.


\(^{17}\) La Monte Young, *La Monte Young and Marian Zazeela at the Dream House: In Conversation with Frank J. Oteri*, 2.
unstroked sound, while Rădulescu invokes them as “self-generative functions”\textsuperscript{18}, one of many sound categories within his works. Hans Zender uses the technique in parallel to other harmonic principles to create a “harmony of opposing tensions”\textsuperscript{19}. Ezra Sims has taken the technique even further. It emerges as a guiding harmonic principle in his 1987 composition, \textit{Quintet}.

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\textsuperscript{18} Horațiu Rădulescu, \textit{The World of Self-Generative Functions as a Basis of the Spectral Language of Music}, 322.

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Analysis I: Ezra Sims’ *Quintet*

Ezra Sims began using combination tones through an intuitive process of composing in Just Intonation.

...a strong - and unwilled - tendency, while composing, to hear the notes in my imagination as if they were related by ideal harmonic ratios and creating (or at least strongly implying) resultant tones - both difference and summation. This seems to happen in the case of both harmonic and melodic juxtaposition. Indeed, the first instance of it that I was forced to notice was melodic, and occurred while I was writing my first string quartet when a melodic succession of an E and an F seemed to demand that the next note be a quarter-sharp E in the octave above. This, I later decided, must have been an instance of a 16th harmonic reacting with a 17th to demand a 33rd.20

Many composers and performers are familiar with more than twelve equal divisions of the octave, using quarter tones and occasionally sixth tones creating 24-tone and 36-tone scales. These tuning systems expand the interval palette substantially and in some cases allow for better approximations of Just intervals. For example, the Just 7:4 can be closely approximated with sixth tones. A more extreme example is the 72-tone equal temperament used by Ezra Sims21 in most of his mature works, including his *Quintet* for clarinet, two violins, viola, and cello. This tuning divides each half step of the chromatic scale into six equal parts with each discrete pitch 16.66 cents apart. When Sims approximates harmonics of a given fundamental there is only an 8.33 cent maximum margin of error. Like the pioneering work of Harry Partch and Ben Johnston, Sims uses an equal-tempered tuning scheme to approximate a Just


21 The same microtonal scale is used by Hans Zender in his works using combination tones, though the symbology is different. See Zender, *Die Sinne Denken*. 
ideal. This is a useful compromise between the ideal of Just Intonation and the flexibility of Equal temperament. It also builds on traditional notation.

...it was necessary to have a division of the whole tone equal to the least common denominator of the fractions, namely 12. This meant a 72-note octave, just as it had earlier been necessary to have a division of the whole tone using the least common denominator of 1 and 1/2, that is, the chromatic 12-note octave, in order to transpose the collection of (ostensibly) equal-tempered whole and 1/2-tones that is the diatonic scale to begin on any member of itself and retain the proper succession of its intervals.²²

Sims notates the 72-note system using the standard accidentals and three additional symbols (Figure 4). Using these symbols in combination, the 72-tone ‘chromatic’ scale²³ is assembled (Figure 5).

Figure 4. Four Inflection Symbols used in Ezra Sims’ Quintet²⁴.
From this, a ‘diatonic’ scale (Figure 6) is assembled consisting of a fundamental/tonic and 1/12-tone approximations of Just intervals from harmonic partials 16-32.

The music uses an 18-note subset of the 72 notes in the same way that tonal music uses the 7-note diatonic subset of the 12. At any moment, there is in effect a transposition of that subset that defines a unique tonal region in exactly the same way transpositions of the diatonic scale do. ... This makes a full 18-note scale made up of a succession of six 1/3-tones, two 5/12-tones, seven 1/3-tones, and two 1/4-tones.

\[\text{Figure 5. The Construction of a 72-tone Chromatic Scale}^{25}.\]

\[\text{The stars indicate the same pitch in alternative spellings.}\]
Such scales were not entirely new at the time of the *Quintet*. Sims cites similarities to Wendy Carlos’ “harmonic scale”\(^\text{27}\), used on her *Beauty in the Beast* album from 1986. This diatonic scale can be transposed to begin on any of the 72 chromatic pitches, just as the traditional diatonic scale can be transposed to any of the 12 traditional chromatic pitches. Sims uses this flexibility to modulate not only to the key area of the *dominant* (Just 3:2) but also to the key area of the Just “lesser undecimal tritone”\(^\text{28}\) (Just 11:8) and other distant, yet related key areas.

Sims discovered a method of enriching tonality by expanding its breadth. The category of relatively consonant intervals is extended to include any pitches that correspond to harmonics 8-15, shown in Figure 6 with open noteheads. He views the other pitches, corresponding to higher harmonics, as less stable, shown in Figure 6 with filled noteheads.

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\(^{26}\) This example is taken from *Yet Another 72-Noter*, page 31.


As in tonal music, it is important here to make a clear distinction between the *fundamental*, analogous to the *root* of a chord, and the *bass*, or lowest note at any given moment in the music, even though they are in many cases the same pitch.

The *Quintet* follows a succession of fundamentals, which are clearly labeled in the score to provide harmonic context for all the pitch material. As is the case for much music in Just Intonation\(^{29}\), the fundamentals are well below the range of the lowest instrument, in this case the cello. The bass pitch at any given time is a relatively low harmonic of that fundamental. Sims considers the change of fundamentals as analogous to a traditional key change\(^{30}\). Their order and relationship to the global tonic are carefully planned. The first movement introduces a sequence of fundamentals designed early in the process of developing the piece\(^{31}\). The sequence is G, D, B\(^{1/12}\)-low, F\(^{1/6}\)-low, A, and C#\(^{1/4}\)-low. These pitches are the 1st, 3rd, 5th, 7th, 9th, and 11th harmonic partials of a global tonic G. At the beginning of the first movement these are heard in measures 1-10. The same progression is then repeated in measures 11-20, transposed to start on C#\(^{1/4}\)-low. The 1, 3, 5, 7, 9, 11 progression is preserved now with a different tonic. The second progression could also be described as 11, 33, 55, 77, 99, and 121 in relation to G. This figure is transposed several more times to other harmonics of the global G fundamental. The first movement, however, is not discussed in depth in this analysis because the approach to combination tones in that movement is the same as in the second movement where it is more fully developed.

\(^{29}\) See works of La Monte Young, Harry Partch, Ben Johnston, Glenn Branca, Kyle Gann, Wendy Carlos, and others.


\(^{31}\) ibid, 3.
The second movement of the *Quintet* provides a clear example of Sims’ use of combination tones in this Just context. The final dyad (B, F) of the first movement is held over as pedal tones in the viola and violin to begin the second movement. The first fundamental of the piece is labeled as ‘E’, meaning these pedal pitches could be labeled as harmonics [6] and [17]. Of course, one could as easily call them [12] and [34] or [24] and [68], but since the lowest possible octave will always provide a simpler solution, they are assigned to the lowest whole numbers which allow all voices to be labeled.

Sims does not define the octave of the fundamental, only stating its pitch class in the score. Because of this, the relative level of harmonic complexity in a given frequency range is not defined. This ambiguity allows for the seamless reinterpretation of pitches into a new octave of the same pitch, by multiplying or dividing by a power of two. For example, if a pitch is defined as the 8th harmonic, Sims can only move up or down by approximately a major second, to the 7th or 9th harmonics. At any time he can choose to move the fundamental down an octave, making the same pitch the 16th harmonic, which allows movement by approximately a minor second, to the 15th or 17th. Or he could transpose further still to the 32nd harmonic, now accessing quarter-tone intervals to the 31st or 33rd harmonics. All of this can be accomplished without changing the pitch class of the fundamental. The amount of harmonic complexity in a given octave can be doubled at any time by transposing the fundamental down an octave.

Sims used the two initial pitches as generators for combination tones in the other voices, assigning the sum tone to the solo clarinet (Figure 7). The second violin
remains constant as pedal, while the clarinet then moves in a melody of harmonics. As the clarinet moves, it maintains its role as sum tone of the other two parts, which creates a need for the lower part (viola and cello) to change. The assignment of the generating tones in Sims’ work is flexible. If Sims composed the clarinet line first, then it is just as easily labeled as the higher generating tone, paired with the second violin. Then the viola/cello line takes on the role of the difference tone. The difference in harmonic (frequency) between the clarinet line and pedal 17th harmonic in the second violin creates the resultant bass line seen in Figure 7. Note that the clarinet is transposed in all the score examples, sounding a major second lower than written.

Figure 7. Quintet, Beginning of Movement II with Annotated Harmonics.

The above method of composing is not so different from a traditional contrapuntal approach in which the movement of one voice requires a resolution in another voice.
However, Sims’ method is much more stringent. He has no direct control over the resultant line, so he makes decisions for both lines at once. The compositional process is similar to writing a canon, where all resultant harmonies must be considered in the construction of a single melody. The chief decision is which note to move. In the cello at the end of Figure 7, the composer was compelled to return to the sixth harmonic (B) to end the phrase, instead of the fifth harmonic (G-sharp), which would have been required by the other two pitches (22-17) on the downbeat. The meant the second violin needed to adjust.

_Immediately, the clarinet wanted to enter on the [sounding] 1/6-high A#, which is the 23rd harmonic, the summation tone of the B and F... This line made me hear the bass line... But, as you will have noticed, the B at the end of the bass phrase implies, with the clarinet line, a resultant not of the 17th harmonic, but the 16th: which made the moment seem to demand an old-fashioned suspension - 17th holding past its generative context, then resolving to the ‘consonant’ 16th harmonic._

This process continues for the rest of the movement, with each pitch harmonically dependent on all the other pitches. In the next three measures (119-121), the process is expanded to include all five parts (Figure 8a). The relationships are consistent, each pitch being either the sum or difference of two other pitches.

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_32 Ibid, 3._
Figure 8a. *Quintet*, Movement II, Measures 119-121.

Figure 8b. *Quintet*, Movement II, Measures 119-121, Harmonic Abstraction.
Figure 8b is a harmonic abstraction of measures 119-121 (excluding the final eighth note), written homophonically to simplify the progression of combination tones. The five instruments are represented by the horizontal rows. The first five chords are the same as in Figure 7, described here as [P], [Q], and [P+Q]. The next five chords are identical with the addition of the [P-Q], and the final chord adds the [2Q-P]. This is one of only a few cases in the *Quintet* in which Sims uses a pitch from the third order of combination tones.

This music illustrates again the interconnections in combination tone systems. Beginning with any two harmonics in the system, one can generate all the others. Furthermore, each pitch contributes to the acoustic resonance of the chord, because every added tone provides further reinforcement of the others through arithmetical combination. The more nodes of the system that are present, the more “connections” can be drawn to reinforce the system\(^3^4\) (Figure 9).

\(^3^3\) The five pitches in the final chord could alternately be labeled (bottom-to-top) [P], [Q], [P+Q], [2P+Q], [2Q+P]. As discussed in the introduction, the arithmetical relationships are more important than the assigned variables.

\(^3^4\) The amount of “connections” (sets of two) can be expressed with the following formula when x=number of pitches: \(x!/2(x-2)\)!
As previously discussed, the fundamental for any given section in the *Quintet* is treated similarly to a key in traditional tonal music. Just as tonal composers would modulate to a related key (dominant, relative minor, etc.), Sims modulates many times to related fundamentals. The first modulation occurs in measure 123 from the global tonic, E, to B (Figure 10). The composer’s choice of the dominant, B, is not a holdover from the tonal tradition, but is a Just 4:3 (inversion of 3:2) the closest relationship of two pitches excepting the octave. Because B is the third partial of the E harmonic series, any harmonic above E that is divisible by three will also belong to the B harmonic series. In the beginning of measure 122 (Figure 10) there is a four-note combination-tone chord above the fundamental E. On the downbeat of measure 123, there is another four-note combination-tone chord, now above a B fundamental. On the last eighth note of measure 122, there is a four-note combination tone chord, spelled as 48:36:24:12 above the E fundamental (Figure 10). Since all of these are divisible by three, the same chord could be written above B as 64:48:32:16. This chord can be more easily understood if reduced by two octaves to be written as 12:9:6:3 in E and
4:3:2:1 in B, but is labeled in the higher register to show its relationship to the surrounding harmonies. Just as any diatonic chord can function as a pivot chord between two related keys in tonal music, so can any common chord between two related fundamentals in this music.

Figure 10. *Quintet*, Movement II, Measures 122-123 with Annotated Harmonics.
Counterpoint

As shown in Figure 7, Sims work bears a resemblance to traditional Sixteenth-century contrapuntal techniques throughout the Quintet, whether consciously or subconsciously. The classification of consonance and dissonance is imperative to contrapuntal technique. In this work, one can understand a combination-tone chord as consonant and all other harmonics which are not combination tones as dissonant. In doing so, one must disregard the accepted standards of consonance and instead favor the internal logic of the combination tone harmonies, which can range from traditionally consonant-sounding (e.g., the 48:36:24:12) to a clangorous bell-like timbre (e.g., the 69:54:39:15).

Composers working with harmonically-driven materials, such as combination tone chords, need to be mindful to avoid excessive parallel homophony, which may sound too blocky. Throughout the Quintet, Sims “minds his P’s and Q’s” to find clever methods of incorporating elegant counterpoint into his harmonic textures. Most of these methods can be easily described using traditional non-chord tone terminology.

In Figure 10, one can identify several non-chord tones. The 52nd harmonic in the clarinet does not fit with the 45:33:21:12 chord (which would have required a 54th or a 57th harmonic) and thus it creates a dissonance. The pitch does not resolve in the following chord but remains until it is recontextualized as a consonant 69th harmonic above a B fundamental. The dissonance could be explained then as an anticipation of the third chord. The 27th harmonic that appears in the viola could be described as a passing tone leading toward the consonant downbeat. In the final eighth note of
measure 123, the second violin resolves from a 54th to a 52nd harmonic, creating a suspension.

The only unexplained dissonance here is the 60:54:8 trichord. The use of a 62nd harmonic in the first violin would have solved this problem, but presumably the composer felt that the escape tone figure in the last two beats of that measure would be more effective with the semitone descent rather than a quarter-tone descent. At the same time, the composer could have moved the 8th harmonic down to a 6th harmonic, but he likely wanted to preserve the dominant-7th arpeggiation in the cello and that change would have created a leaping line. Presumably, Sims wanted to avoid similar motion between three voices on that eighth note and instead opted to hold the second violin’s 54th harmonic as a suspension which would resolve to a 52 harmonic. Interestingly, the resolution does come, but the pitch is recontextualized as part of a 66:52:14 trichord, instead of the expected 60:52:8 trichord.

As was demonstrated in the introduction, there is a disparity of exact intervals between parallel harmonics due to the logarithmic nature of our perception of pitch. Parallel motion in Just Intonation implies two voices moving in the same direction, by the same amount of harmonics. Sims employs this approach beginning in measure 129 (Figure 11).
In this section, the composer keeps the clarinet and the viola in parallel motion in order to synthesize a consistent difference tone in the cello line. With the exception of the few moments when the cello changes pitch, the clarinet and viola maintain the same distance from one another and move in the same direction by the same number of harmonics. In this register (harmonics 29-48), the interval maintained between the two parts ranges somewhere between a major 3rd and a tritone, and has a smoother sound than the exact parallel intervals with which we are accustomed. In a higher register, keeping the same cello difference tones, the intervals between the clarinet and viola would be smaller. In a lower register the intervals would be much larger. This could be
described as *logarithmic parallel motion*, which adjusts for the non-linearity of the human auditory system and Just Intonation.

**Implicit Combination Tones**

Another example of parallel motion occurs in the third movement. In this case, the violins play in close parallel motion for the opening of the movement. In measures 158-159, for example, both parts alternate between two harmonics of a C# 1/4-low. The first violin alternates between harmonics 23 and 24, whereas the second violin alternates between harmonics 21 and 22 (Figure 12).

![Figure 12. Quintet, Movement III, Measures 158-159.](image)

As in the previous example, the two parts maintain a consistent distance from each other, in this case two harmonics. Unlike the previous example, the difference tone is not played outright. The 2nd harmonic would be a C# 1/4-low three octaves and a fifth below the violins at the bottom of the piano, unreachable by the cello. Sims remarked about this section of music that “the consistent use of parallel seconds,
however, must no doubt have the effect of suggesting the tonic...” This is a more subliminal use of a difference tone pedal than the example from the second movement, here the difference tone is intended to be synthesized in the listener’s ear as a *terzo suono*, instead of being performed outright in another instrument.

In the fourth and final movement of the *Quintet*, measures 248-252, Sims takes yet another approach to deriving harmony from combination tones. In this section, he creates a homophonic chord progression in which each harmony is derived from a different fundamental, using the original progression (1, 3, 5, 7, 9, and 11 of G) from the beginning of the piece. Up to this point, most adjacent chords in this work have been comprised of different harmonics from the same fundamental. Chords culled from different fundamentals provides a new challenge to the composer, even though the fundamentals are harmonically related.

Though none of these chords themselves are composed of combination tones, Sims used the difference tones of each vertically adjacent pair of pitches to guide the progression (Figure 13).

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This low chorale is never played outright by the instruments, but is implied. Sims’ states “[It] seem[ed] to help if the adjacent notes of these complex chords related in such a way that, in isolation, they might produce difference tones that would, in the aggregate, form clear and simple chords. It seemed further desirable that those implied resultant chords should relate smoothly and directionally, if the actual ones were to do so…”

Composing a microtonal chorale is a tricky business, especially when the harmonies are only logical insofar as their fundamentals are harmonically related. So,

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36 This example, in the composer’s own hand, was also taken from Ezra Sims, *Harmonic Ordering in Quintet: A Use of Harmonics as Horizontal and Vertical Determinants*, 4.

by carefully controlling the voice-leading of the difference tones, Sims found he could ensure a healthy mix of contrapuntal motion types and reinforce the smoothness of the sounding chord progression. In a way, this is simply an extension of the original application of combination tones used by Tartini. Here, instead of helping tune Just intervals, they are used to compose a more complex chorale. This section of music is possibly the most complex, and yet surprisingly consonant, harmonic progression in the piece. It is a fitting ending.

**Quintet Summary**

Ezra Sims employs combination tones from the lowest two or three orders in several different harmonic schemata in his *Quintet*. He uses a contrapuntal method similar to traditional Sixteenth-century species counterpoint. The paradigm alteration in his counterpoint as compared to traditional counterpoint hinges on the interpretation of consonance and dissonance. In species counterpoint unisons, thirds, perfect fifths, sixth, and octaves are considered consonances while seconds, fourths, diminished fifths, and sevenths are considered dissonant. In the *Quintet*, the intervals from the bass are not the defining factor but rather, the arithmetic relationships between voices, which are quantized to harmonics of a given fundamental. The combination tones are consonant, all other harmonics are dissonant.

Sims controlled the fundamentals in the work by selecting a global tonic, G, and choosing a progression of new fundamentals from closely related (i.e. relatively lower) harmonics of the global tonic. Because of this hierarchy, every pitch in the piece could
be named as a harmonic of the global tonic of G. One would simply find the harmonic in relation to the notated fundamental: the initial pitches in Movement II (Figure 7) are harmonics six and seventeen. Since E is the 27th harmonic of the global tonic G, these pitches could be called 162 and 469 respectively. This is less elucidating than the analysis in E and requires a fantastically low fundamental G 1.5 Hz, more than four octaves below the bottom of the piano. The relationship, however extreme, provided Sims with a frame of reference for how far away from the global tonic he wandered at any given time.

After composing generating tones in certain instruments, in most cases Sims used the combination tones directly as the pitches in the remaining instruments, as in Figures 7-11. In other cases, however, Sims merely suggested the difference tone, allowing for its spontaneous creation in the inner ear of the listener, as in Figure 12. At first this may seem quite a stretch for the composer, but we must remember that this phenomenon provided the initial impetus for Sims to work with combination tones in the first place. His intuition guided him to compose the initial sum tone in the second movement. It is only fitting that at some point the listener should also be invited to perceive the phantom tones without them being explicitly performed. In the most extreme case, in Figure 13, Sims used the combination tones as a compositional guide creating an implied harmonic structure of difference tones that is not meant to be perceived.

Ezra Sims has developed a tonal language in his works using Just Intonation, approximated by a 72-tone Equal Temperament. In his Quintet, the harmony is controlled by combination tones between harmonics of a series of related fundamentals.
The work was composed carefully so that the pitches maintain combination-tone relationships giving an inherent logic to the harmony of the piece.
Analysis II: Ben Hjertmann’s *Angelswort*

My composition, *Angelswort* (2012), is a seven-movement recorded work following a dream-like, mythological narrative. The piece is scored for voices, saxophone, viola, electric bass, piano, electronic organ, and sampled sounds. The harmonic language of the entire work is structured using combination tones. For the sake of clarity and brevity, only two movements are explicated in the following analysis, *Passacaglia: L’Homme Armé* and *Chorale: Angelswort*. All of the melodic and harmonic combination-tone techniques used in the larger work can be explained using the techniques described in these two movements.

**Passacaglia: L’Homme Armé**

*L’Homme Armé* is a short movement for three male voices, soprano saxophone, viola, organ, and piano. The original *L’Homme Armé* is a French secular song dating around the turn of the 15th century that became one of the most popular cantus firmi used in polyphonic masses in the Renaissance and continues to be used today. Famous settings include those by DuFay, Ockeghem, Josquin, and Palestrina. The text “the armed man should be feared...” was likely intended to create an allegory, but in my piece the text is interpreted literally, contributing to the larger narrative. The

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character of the music is evocative of a Renaissance secular song and is also described as a *passacaglia*, because it is in a triple meter and has a repeating bass line.

The *L'Homme Armé* melody is used as a *cantus firmus* ostinato in lowest register of the piano. The piano is hidden in the recording using a tight, high-frequency band-pass filter. It is played at a low volume in order to provide a quiet impression of the cantus firmus melody. *L'Homme Armé* uses two tuning systems simultaneously. The piano is performed in 12-Tone Equal Temperament, maintaining a consistent ground. The vertical harmonies created above each chord are tuned in an idealized Just Intonation in which each pitch is tuned to harmonics of the piano line. Therefore, the horizontal (melodic) intervals in any given part are in many cases not defined by a single tuning scheme. They can only be explained vertically (harmonically) in relation to the current fundamental pitch.

The notation system contains quarter tones and sixth tones in addition to traditional accidentals (Figure 14).
Like Sims’ *Quintet*, the vertical sonorities are composed in Just Intonation. Here, the accidentals are not as specific, and are rounded to the nearest sixth tone or quarter tone to simplify the written notation. However, neither the accidentals, nor the natural written pitches adhere to any equal-tempered scale. Because the *L’Homme Armé* movement was intended for recording, the players performed along with perfectly-tuned synthesized recordings of their lines. No artificial *ex post facto* tuning effects (e.g., autotune) were used on any of the recordings. With the exception of the organ, all lines were executed by human performers. Since the guide-tracks were part of the initial conception of the piece, using the above tuning system was logical. Performers who are not well-versed in the twelfth-tone system of Sims may find this notation more comprehensible because it more closely reflects the common accidentals.

Unlike Sims, I placed the fundamentals in a particular register, two octaves below the piano bass line, to preserve the melodic contour of the *cantus firmus*. This allowed for a greater variety of harmonics that would be playable by the instruments, and therefore, a larger variety of combination-tone harmonies. It is worth noting that while the fundamentals themselves fall below the range of human hearing, the remaining (higher) spectrum of a sound within this range is certainly audible.

Part of the rationale behind the use of harmonics in this movement is that it creates an artificial formant of the *cantus firmus*. Every pitch we hear in this movement would already be present in the *L’homme armé* melody if it were played in this extreme low register. Therefore, all the instrumental and vocal lines in the movement could be
seen as a heavily-filtered, and amplified recreation of the fundamental bass, which is otherwise inaudible. In a sense, it is a vertical canon that uses a single melody to derive a much larger work, following a strict harmonic rule.

Like Sims’ Quintet, L’Homme Armé uses harmonics that are combination tones to create harmonies. Unlike the Quintet, L’Homme Armé includes all the pitches from the third order of combination tones discussed in the introduction, and unlike the Quintet uses only combination tones for all the harmony in the movement, not just the consonances.

To begin composing, I constructed a harmonic limit of combination tones chosen from the lowest three orders, that is, $P$, $Q$, $P\pm Q$, $2P\pm Q$, and $2Q\pm P$, eight pitches in all. Intrinsic to each of those combination tones is a relationship to the motion of generating tones. As in species counterpoint, the motion of the generating voices can be contrary, oblique, similar, or parallel. Figures 15a and 15b demonstrate the relationships between motion types of the combination tones in the lowest three orders. In this example, A110 Hz is used as a fundamental to demonstrate a Just context. However, the same motion types would apply with any tuning system used between the two generating voices. Figure 15a demonstrates two examples of oblique motion between the generating voices. Note how the direction of motion in $[Q-P]$ and $[2Q-P]$ remains consistent while the rest change between the first and second halves of the example. With oblique motion between the generating voices, only one or two (depending on which generating voice moves) of the combination-tone voices move in contrary motion while the rest move in similar or parallel motion.
Figure 15a. Lowest Three Orders, Generating Voices in Oblique Motion.
As stated in the analysis of Sims’ *Quintet*, parallel motion outside of an equal-tempered system takes on a different meaning. Parallel motion between harmonics implies a consistent *distance* between the harmonics. In the first half of Figure 15b, the generating voices move in this way. With parallel motion between the generating tones, all of the combination tones move upward, except the difference tone which remains static. In a texture with six voices, parallel motion between most or all voices quickly becomes monotonous, sounding similar to an electronic harmonizer. Similar motion is largely the same except the difference tone is not static. Both parallel and similar motion were avoided between the generating voices in *L’Homme Armé*.

Contrary motion between the generating voices, as exhibited in the second half of Figure 15b, provides the greatest variety of motion types in the combination tones. If the voices move by the same number of harmonics in either direction (here moving by one harmonic) then the sum tone remains static. Four voices move up, two down, one remains, and one, [2P-Q], changes direction. Because of the greater variety of motion in the combination tones available, contrary motion was preferred between the generating tones in the piece. The same preference also exists within traditional counterpoint.

In each of these examples a pedal tone was generated in one of the voices. As in traditional counterpoint, this pedal gives a sense of grounding useful in an otherwise tumultuous microtonal texture. Of course, one can compose in similar or contrary motion without maintaining a consistent interval between the parts, and thus not creating a pedal tone. The resultant motion is mostly the same.
Figure 15b. Lowest Three Orders, Generating Voices in Parallel and Contrary Motion.
In orchestrating *L'Homme Armé*, the generating tones, P and Q were assigned strictly to the tenor and baritone voices throughout the movement. There are six remaining combination tones and only four remaining voices in the movement (disregarding the hidden piano): bass voice, soprano saxophone, viola, and monophonic organ. The disparity is intentional. Two pitches from the second and third orders were left out of each chord, which allowed me as the composer some control over the voice-leading. Since the combination-tone assignments of each of the four accompanimental voices was fluid, the inherent motion types described in Figures 15a and 15b could be avoided when the instruments/voices switched combination-tone assignments. This created an opportunity to eschew similar and parallel motion when it was not desired.

Figure 16 displays the six performing instruments’ parts as well as the six combination-tone lines and the piano continuo, which can be used as a reference for the fundamental pitch in measures 9-12 of the movement. This elucidates the orchestration process in *L'Homme Armé*.
Figure 16. *L’Homme Armé*, Measures 9-12, Orchestration and Third-Order Pitch Palette.
For the most part, the above rules were strictly followed in composing the movement, but in some cases certain liberties were taken in order to smooth the melodic lines and overall counterpoint. For example, the bass voice holds a suspension between measures ten and eleven. In measure nine, the soprano saxophone uses harmonics 88 and 96 as an escape-tone gesture between 80 and 128. In both cases, as in the Sims, the combination tones are treated as chord tones, while 88 and 96 are treated as non-chord tones. The two non-chord tones create a line with small integer harmonic ratios to the fundamental, so the result will not sound shocking. The measure could be reduced by three octaves to 10-11-12-16.

I began with the tenor melody, which was freely composed from the available harmonics. If we allow a range of two octaves for the voice, A2-A4, in measure 9 above an A fundamental, then it has access to harmonics 16-64, 49 distinct pitches. That is double what the same range would accommodate in 12-Tone Equal Temperament, 25 pitches. Because of the logarithmic nature of the harmonic series, the lower octave contained half the possibilities of the higher octave, and some intuitive pitches are decidedly missing from this harmonic palette. Most notably, a traditional “Fa” is missing in the harmonic series. A perfect fourth is a 4:3 ratio in Just intonation, so in most Just systems, the scale is built as a 4:3 above “Do”, implying a “Fa” fundamental. However, if working strictly within a stationary harmonic series, this fourth is between “Sol” and “Do”, not “Do” and “Fa”, and one must use a very high harmonic in order to approximate the “Fa”. In the lower octave of the tenor’s range, the 21st harmonic is closest, but is about a sixth-tone lower than the equal-tempered “Fa”, as it is the 3rd harmonic of the “flat” 7th harmonic. In the upper octave, the 43rd harmonic is closer, being only about a
12th-tone higher than the equal-tempered “Fa”. This draws yet another parallel between Sixteenth-century species counterpoint in which perfect fourths above the bass were considered dissonant.

Once the tenor line was complete, a contrapuntal voice was composed for the baritone. The composition of these two parts determined the harmonic palette for the entire ensemble. The next stage in the process was calculating the frequencies of the six combination tones, seen in Figure 16 on staves 7-12. Then I made an initial attempt to construct a viable organ part. Since the piano is almost inaudible, the organ is the lowest instrument in the ensemble and the de facto bass line. This proved to be the most difficult task in composing the piece. Because of the logarithmic nature of pitch, as discussed in the introduction, minute changes in the generating tones [P] and [Q] cause large changes in the difference tones. This is another reason why it was necessary to have both the difference tone [Q-P], and the cubic difference tone [2P-Q] available to construct a suitable bass line. The large leaps in each of the difference tone voices, seen in Figure 16 on staves 11 and 12, attest to this problem. In many cases, one of the generating tones needed adjustment in order to smooth the organ part. I found that I needed to limit the movement of one of the generating voices for this purpose. I continued this process building the chords from the bottom up, moving to the bass voice, then the viola, and finally the soprano saxophone.

Then, I tested the homophonic chord progression with microtonal software39. After a final version of this progression was complete, I created polyphonic counterpoint in a

39 The microtonal harmonies were tested with the “Little Miss Scale Oven” software and some of my own Max/MSP patches in conjunction with a MIDI controller.
traditional manner using passing tones, neighbor tones, anticipations and suspensions culled from the unused combination tones.

As seen in Figure 16, the baritone’s A is labeled with three different harmonic numbers. As the piano fundamental changes, a pedal tone is recontextualized as a new harmonic. The frequencies of each of these harmonics differ slightly. They are 440.0 Hz (32\textsuperscript{nd} of A), 440.5 Hz (24\textsuperscript{th} of D), and 441.5 Hz (27\textsuperscript{th} of C). The variation is equivalent to a two-cent change for the first transition and a four-cent change for the second, both of which are virtually indistinguishable to the ear.

Combination tones are also used melodically in \textit{L’Homme Armé} in some places. Figure 17 shows a clear example of melodic combination tones. In this example, the tenor voice sings three harmonics above an A fundamental. The two harmonics (30 and 34) create a sum tone (64) which is sung on the third beat of the measure. Because they are not sustained simultaneously, they will not reinforce the acoustical combination tones already present, except slightly through reverberation. However, the linear combination tones are analogous to an arpeggio of a triad, which is not the same as a sounding chord, but nonetheless implies it. Furthermore, hearing this structure horizontally may serve to illuminate it for the listener, since it appears many other times vertically. In the seventh movement of \textit{Angelswort}, I expanded this melodic concept.
Perhaps the most fascinating element of this system of harmony is its relationship to tonal music. In *L'Homme Armé*, a middle ground was sought between the clangorous sonorities of clustered harmonics, and the perfect consonances of simple Just Intonation. It was my intention to create indisputable “tonal” cadences in this movement, but using a method that would place them in a new context. Figure 18 excerpts a cadence from the piece, which is not unlike a traditional Imperfect Authentic cadence.
There are only two oddities in this example that would not have appeared in traditional tonal music: the flat-7 appearing in the first chord (creating a V7/IV - V7 - I progression, instead of I - V7 - I), and the improper resolution of the sevenths in both penultimate chords. Besides these oddities, it is remarkable that using a harmonic technique as new and as strict as this could result in such a familiar tonal cadence. In fact, familiar tonal chords are quite common in the collection of possible combination tone chords.
The prevalence of familiar chord progressions in *L’Homme Armé* can be attributed to two factors: simple bass relationships, and simple ratios between generating tones. The “tonality” of the combination chords themselves is a product of simple relationships between the generating tones. In Figure 18, the ratios of the generating frequencies are as follows 4:3 (of A), 5:4 (of E), 2:1 (of A). (Make note that this is a rare exception in which [P] is reassigned to the bass voice instead of the baritone.) These are reduced (transposed down two octaves) from 32:24, 20:16, and 32:32 respectively. The distance between the two numbers in the ratio accounts for the interval between them, but the familiarity of these intervals is attributed to their relatively low prime factors. The most familiar sounding intervals and chords are those constructed from whole numbers with prime factors of five or less. This is referred to as 5-Limit Just Intonation. La Monte Young explains, “It’s interesting if we look at the history of Western classical music. If we were to tune it in Just Intonation, it would all be factorable by 2s, 3s, and 5s: 2s being octaves, 3s being 5ths, and 5s being the major 3rds.” The lower the prime factors in the generating interval, the simpler and more familiar the resultant combination tone chord. The most extreme example would be one in which the generating interval is a perfect unison, or 1:1, as would have occurred in the last measure of Figure 18 if P had not been reassigned. If P=1 and Q=1, the set of pitches produced in the first three orders would be: P+Q=2, P-Q = 0, 2P+Q=3, 2Q+P=3, 2Q-P=1, and 2P-Q=1, for a chord of 3:2:1, which sounds like an octave and a perfect fifth. P was reassigned to the bass.

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40 Note that in measure 41 the generating tone P was relocated from the baritone to the bass voice.

41 James Tenney, *A History of Consonance and Dissonance*.

42 La Monte Young, quoted from *La Monte Young and Marian Zazeela at the Dream House, In Conversation with Frank J. Oteri*, 58.
to use the generating interval of 2:1, which creates a full major triad, 5:4:3:2:1, which was preferable in this context.

The other factor contributing to the familiarity of the progression is the melodic interval sequence of the fundamental bass line. Just as with the ratios of the generating tones, the melodic ratios between successive bass pitches contributes to the perceived “tonality”. In the case of *L’Homme Armé*, there is a prescribed bass line from a tonal *cantus firmus*, but in any other case, the same guiding principles would apply. In Figure 18, the ratios of the consecutive bass pitches are: 3:2 and 3:2 again returning to the original pitch. Because 1:1 and 2:1 creates no cadence at all, this is the lowest prime factor ratio for cadential bass pitches, and therefore, the simplest cadential progression.

It is certainly no coincidence that the lowest prime factor bass motion creates all of the most ubiquitous cadences in Western music — 3:2, the Authentic cadence, retrogradated to 2:3, the Half cadence, and inverted to 4:3, the Plagal cadence.

The combination tone method of harmony and counterpoint used in *L’Homme Armé* draws unique connections between the so-called “spectral” school of composition and the traditional counterpoint and harmony of Fux and Rameau. The staple triads and seventh chords of Western music appear in progressions alongside clangorous complexes of harmonics. The confluence of traditional and contemporary harmonies in this unified context allows an auditor to perceive traditionally consonant and dissonant sonorities as similarly constructed, existing on a continuum of harmonic complexity. By extension, this continuum might provide a new vantage point from which to examine other works of the recent or distant past.

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43 François Rose, *Introduction to the Pitch Organization of French Spectral Music*. 
Chorale: Angelswort, and the ‘Golden Chord’

The fifth movement of the piece is the titular movement, Chorale:Angelswort. It is composed for lead voice, a chorale of voices, and rhythmically triggered sound samples that are excerpted from other movements of the larger work. The lead voice and triggered samples are only loosely related in pitch to the chorale. This movement employs a markedly different approach to combination-tone harmony in the chorale, approaching a *Golden Chord*. In order to explain the harmonic organization in Chorale:Angelswort, the phenomenon of the Golden Chord must first be explicated.

If one begins by taking the sum of a unison and deriving the second harmonic, then adds the second harmonic to the first, then adds the third to the second, then the fifth to the third, etc., always taking the sum of the previous two harmonics, one generates what can be called a *Fibonacci Chord*. This chord can be described as a filtered harmonic spectrum in which the only partials present are the Fibonacci-numbered harmonics: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, etc., which will be shown in Figure 24.

In mathematics, it is demonstrated that the ratio between adjacent members of the Fibonacci Series converge upon the golden ratio, which is about 1.6180339...\(^4\) It can be extended in the other direction as well, and the ratio between 0.610339... and 1 is the same as the ratio between 1 and 1.6190339... This same phenomenon occurs

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\(^4\) Both the Fibonacci Series and the Golden Ratio are recurrent themes across the arts for centuries. These phenomena appear naturally in sunflower seeds, sea shells, and in many other places. In the visual arts, the Golden Ratio has long been supposed to lend pleasing proportions to a work. In music, Bartok, Rădulescu, and others have composed using the Golden Ratio to guide formal, harmonic, and rhythmic motifs. See Robert Lawlor’s *Sacred geometry: philosophy and practice.*
between the Fibonacci harmonics. We notice that the distance between two adjacent Fibonacci harmonics converges on 833 cents.

If we begin with A 440 Hz and multiply it by 1.6180339 we get F 711.93491 Hz. This F is 833.09 cents above the A. This process can be continued up or down from any frequency, and a scale can be created in which every step is 833 cents, the *Golden Scale*, or if stacked harmonically could be called the *Golden Chord*. Because of our logarithmic perception of pitch, the ever-expanding distances in frequency between adjacent tones moving up the scale result in a perceived linear scale/chord. 833 cents is a sixth-tone larger than an equal-tempered minor sixth. Three adjacent tones of the Golden Scale create something quite close to a second inversion minor triad, which could be called a *Golden Triad* that repeats up two octaves and a minor second (Figure 23).

![Figure 23. Series of Three Golden Triads.](image)

Since all the intervals are equal, this scale shares a similar disorienting quality with the Whole Tone Scale, though the interval is four times larger. There are two distinct Whole Tone Scales in 12-Tone Equal Temperament. Since the Golden interval requires the use of sixth tones, there are 25 unique Golden Scales in 36-Tone Equal Temperament. Similarly, a Golden Chord with 26 nodes (spanning 18 octaves) would
be required before a pitch would be repeated. Even then, the one cent that had been
rounded off for each Golden triad (833+833+833=2499 rounded to 2500 cents) would
add up to about 8.5 cents, a significant deviation. Since humans only perceive about
ten octaves as pitch, it is easier to say that the scale contains no duplicate pitch
classes.

The interval content of the Golden Chord starkly contrasts the harmonic series.
The Golden Chord is equally spaced in all registers and there is only one extant interval.
The harmonic series contains an infinite number of intervals which shrink as one
ascends the series, therefore creating an enormous disparity in the pitch density of
different registers of the same series. Any two adjacent pitches in the Golden Chord
can generate the rest of the chord through progressive addition or subtraction. Unlike
many other combination-tone harmonies, the generating tones are unimportant.

In contrasting the Fibonacci Chord with the Golden Chord, distinct differences can
be observed. The chords in Figure 24 are calibrated to possess an A 440 Hz in
common so as to illuminate their differences. The upper tetrachord in both harmonies is
approximately identical whereas the bottom tetrachord is quite different. Because the
Fibonacci series begins with small integers their approximation of the Golden interval is
quite poor at first, but as the numbers get higher and contain more digits, they begin to
more accurately approximate the Golden Ratio. This disparity could be removed by
beginning on a higher node of the series, for example, 34, 55, 89, 144, 233, 322, etc.
By contrast, the Golden Chord is accurate to the Golden Interval throughout all
registers.
Hans Zender, a composer who works frequently with combination tones, discusses this phenomenon in his article “Gegenstrebig Harmonik.” Zender states that one can approach the Golden Chord by beginning with any interval, small or large, and adding the sum of the previous two pitches (Figure 25). “...so we could continue to always add the two highest resulting tones ... the peculiar result of this operation carried out in the different intervals on our list, indicates that all the resulting ‘spectra’ approach the interval proportions of the ‘golden ratio’.”\textsuperscript{45} The process is simple: begin with two tones and add them together, then continue to add the highest tone to the tone just below it. These can be referred to as “progressive sum tones”.

\textsuperscript{45} Hans Zender, Die Sinne denken, Texte zur Musik 1975-2003. (Page 122)
Another way to summarize the progressive sum tones is by using higher-order combination tones. Even though they are not harmonics of a common fundamental, as in the L’Homme Armé, the progressive sum tones could be calculated as harmonics of the generating tones, as explained in the introduction of this document. Looking at the frequencies in Figure 25, the first two tones in either example could be called P and Q. The third tone is [P+Q], the fourth is [(P+Q)+Q] or [2Q+P], the fifth is [(P+Q) + (2Q+P)] or [3Q+2P], the sixth is [(2Q+P) + (3Q+2P)] or [5Q+3P], the seventh is [(3Q+2P) + (5Q+3P)] or [8Q+5P], and the eighth is [(5Q+3P) + (8Q+5P)] or [13Q+8P]. Each of the sum tones in a progressive sum set could be described as higher-order combination tones of the initial generating tones using adjacent Fibonacci harmonics. Therefore, the combination tone orders are less relevant in this context, because only the four

![Figure 25. Sum Tones Approaching Golden Interval (833 cents) Using Contrasting Generating Intervals.](image-url)
particular combination tones involving Fibonacci numbers in an order are used. For example, in the third order, all four pitches are used: [2P+Q], [2Q+P], [2P-Q], and [2Q-P]. However, no pitches from the fourth order can be used, because they cannot contain adjacent Fibonacci harmonics. In the fifth order, only [3P+2Q], [3Q+2P], [3P-2Q], and [3Q-2P] can be used, not [4P+Q] or [4Q+P]. The sixth and seventh orders contain no usable combination tones, and the eighth order again only allows four tones: [5P+3Q], [5Q+3P], [5P-3Q], and [5Q-3P]. Not only are the adjacent Fibonacci harmonics exclusively used in combination tones in the Golden Chord, but they can only be found in Fibonacci orders.

Claude Vivier was wont to harmonize melodies with these progressive sum chords, which he referred to as couleurs. In Lonely Child, for example, he uses the soprano A4 and the G2 as generating tones and adds the frequencies of the top two pitches progressively, generating five higher pitches (Figure 25). As discussed, the seven pitches approximate a Golden Chord. Unlike Sims, Vivier does not place these pitches in a larger Just Intonation context\(^{46}\), though one could interpret them over a fundamental G as 92:57:35:22:13:9:4, which elucidates the similarity to 89:55:34:21:13:8:5, part of the Fibonacci Chord. The distinction is subtle, but necessary. Vivier’s chord is always calculated from two 12-Tone Equal Temperament pitches, which result in combination tones that do not belong to any particular tuning system.

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\(^{46}\) Bob Gilmore, On Claude Vivier’s Lonely Child, 66-78.
Figure 26. Claude Vivier's *Lonely Child*, mm. 24-28, Fibonacci-Type Progressive Sum Tone Chord
Chorale: Angelswort takes this to a further extreme. Like Vivier’s practice, movement V begins and ends with 12-Tone Equal Temperament chords, which are extensions of the final chord from movement IV and the first chord from movement VI, respectively. All of the pitches in the movement are derived from the combination tones of the first harmony, and are unbound to any linear or over-arching tuning system. They drift away from 12-Tone Equal Temperament immediately.

Figure 27. Angelswort, Movements IV-VI, Harmonic Reduction.

Figure 27 is a reduction of the transitions between movements IV, V, and VI in Angelswort. There are two pitches in common with the movements on either side of movement five. These were used as generating frequencies that resulted in the combination tones that comprise the rest of the chords in measure two and four of Figure 27. The initial generating interval is larger than an octave, which results in a difference tone that is higher than one of the generating tones. By the time the Golden
Chord occurs in movement V, the lower generating tone has risen higher, within an octave of the higher generating tone, creating a difference tone which is lower than the generating tones. This causes the voice-crossings notated in Figure 27.

The Golden Chord shown in Figure 28 occurs in *Chorale:Angelswort* at the Golden Ratio division of both the movement and the entire composition. The Fibonacci Chord has a finite beginning with the first partial. The “pure” Golden Chord is limitless and equal throughout all registers. In *Chorale:Angelswort*, the notated harmony contains 21 nodes, 15 of which are perceived by humans as pitches while the other six are heard as rhythms. Figure 28 illustrates these 21 nodes as they appear in the piece, calibrated to A 440 Hz.
The rhythmic layers in Figure 28 were used to amplitude-modulate the pitch layers. The rhythms are not articulated by any of the instruments, but rather as rapid amplitude swells in the higher nodes of the Golden Chord, just as was demonstrated in Figure 1 in the introduction. The combination tones created through amplitude modulation of existing adjacent nodes in the Golden Chord do not create any new pitches, but only
reinforce the other nodes of the chord. This phenomenon is unique to the Golden Chord: no other extant harmony is recursive in this way. Because of this fact, the Golden Chord is a logical endpoint for the combination tone phenomenon; both suggest one another.

Beyond the notated nodes in Figure 28, nine more can be perceived between the durations ranges of rhythm, phrasing, and formal divisions of the movement. These range from the speed of the half note ostinato to the distance from the beginning of the movement to the Golden Chord. Beyond those, there are five more formal divisions that correspond loosely with climaxes in the previous movements, and one final node equaling the duration of the entire work, 26.9 minutes. These 36 total nodes of the Golden Ratio provided an internal harmonic, rhythmic, and formal limitations that proved fruitful in the process of composing the work.

In Sims’ Quintet all the instruments were assigned any harmonic of the given fundamental and combination-tone chords were treated as consonances. In L’Homme Armé there are two generating voices and four other instruments that were freely assigned one of six combination tones. In composing this movement, I only limited my choices in assigning the instruments to the combination tones by attempting to create a singable melody for each part without excessive leaps. In Chorale:Angelswort, each of the seven voices in the chorale maintain their relationships to one another throughout. Because the piece converges on the Golden Chord, it was necessary to maintain the roles in the harmonic structure so that each voice would smoothly transition into and

47 A final node was calculated in the light spectrum (not adjacent, but many nodes higher) as a yellow (FFCC00), which was included in the artwork for the movement.
away from the chord. It is my hope that the listener will intuitively expect the Golden Chord, given that it is implied by the relationships between the voices, and therefore perceive this cadence as a logical arrival.

**Figure 29. Chorale: Angelswort, Singable Frequencies.**

In Figure 29, the singable frequencies for the entire chorale are displayed. With the exception of the first two measures, each voice can be viewed as the sum of the two voices below it and the difference of the two above it. In the first two measures, the
bottom pitch is the absolute value of the difference of the two above it, but cannot be added to the voice above to create a sum tone on the third staff. This illuminates the importance of distinguishing the generating tones, seen here on the middle two staves. Because two pitches were retained as common tones between movements IV and V, they were treated as generating tones for the initial chord of the chorale. In measures 3-12, any two adjacent pitches could be considered generating tones. In the first two measures, the chords can only be generated by the pitches in the middle two staves.

The reason for this is that the generating interval is larger than an octave. The interval of the octave receives no special significance in this context, but the ratio of the octave, 2:1 is the dividing point for voice-crossings. If the generating interval is less than an octave, the ratio between the two pitches is less than 2:1 meaning that the difference tone will be between zero and the bottom generating pitch. The structure of progressive sum tones is maintained. If, on the other hand, the generating interval is larger than an octave, as in measure one and two of Figure 29, the difference tone will be larger than the bottom generating tone that disrupts progressive sum tone structure.

**Angelswort Summary**

The *L’Homme Armé* movement of the larger work *Angelswort* uses a fixed set of combination tones with a somewhat flexible instrumentation. The tenor and baritone voices were treated as [P] and [Q], while the bass voice, soprano saxophone, viola, and organ were freely chosen from the remaining six pitches in the third order. The fundamentals, which are never explicitly heard, are the *L’homme armé* cantus firmus,
which repeats four times over the course of the movement two octaves below the range of the piano. The composition of the tenor and baritone parts controlled the available pitches for all the other instruments, as well as the implied directions of motion in the other voices. Therefore, composing those two voices was a process of trial and error.

The relative complexity of the interval between these two voices, by extension, controls the consonance of the entire vertical structure. In Figure 16, we can observe this phenomenon. At the end of measure ten, the simplest possible generating interval is used, 1:1. This generates a 3:2:2:1:1 combination-tone chord, with a highest prime harmonic of only three. In measure nine of Figure 16, the generating interval between the tenor and baritone voices is a 3:2. Because the interval is relatively consonant, possessing low prime factors\(^{48}\), the combination tones are also relatively consonant. The 3:2 interval results in a 8:7:5:4:1:1 combination-tone chord. The highest prime factor in the generating interval is three, and the highest in the resulting combination-tone chord is seven. At the beginning of measure ten there is a more complex interval of 17:12 between the tenor and baritone, which generates a 46:41:29:22:7:5. The highest prime in this generating interval is 17, and the highest in the combination-tone chord is 41. The results is a much more complex, dissonant harmony. In the *L’Homme Armé*, the relatively simple, consonant harmonies are used as arrival points for all the cadences, and the complex, dissonant harmonies are used in the center of phrases.

The *Chorale:Angelswort* movement was constructed around seven voices, which have a fixed combination-tone relationship in which each voice is the sum tone of the

\[^{48}\text{Ludger Hofmann-Engl, *Consonance/Dissonance - A Historical Perspective* (page 853) interpreting Leonard Euler’s *Tentamen novae theoriae musiceae ex certissimis harmoniae principiis dilucide expositae.}^\]
two voices immediately below. This technique produces the same relationships as in the Fibonacci Chord, a harmony comprised of the harmonic partials that correspond to Fibonacci numbers (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc.) At the moment of the Golden Ratio in the movement (61.8339% into the piece), the intervals between each of the adjacent voices converges to exactly 833 cents, creating an equally spaced Golden Chord. For this climax, additional voices were synthesized with sine tones above and below the singable chorale to extend the Golden Chord throughout the range of human hearing and further into the realm of tempo, rhythm, and form. In this movement, none of the constituent pitches are related harmonically, nor as members of any scale. Each pitch is generated using the frequencies of those immediately below it, without conforming to any tuning system.
In *L’Homme Armé*, I explored the relationship of combination tones to Western tonality. Each phrase ends in a traditional Half or Authentic cadence, but the harmonic progression leading there is markedly different. Similarly, in the music of Ezra Sims a 72-tone tempered notation is used to approximate harmonic partials of low fundamentals. In his music, Sims creates a tonal hierarchy of key areas based on their distance in the overtone series from a global tonic. In Sims’ system, a move from the tonic key area to the dominant key area is just as logical as it is in traditional tonal music.

Both Sims and I have employed a contrapuntal technique remarkably similar to traditional Sixteenth-century species counterpoint. Both use passing and neighbor tones, suspensions, and a balance of motion types. These contrapuntal methods require a clear definition of consonance and dissonance, which are different between Sims’ piece and my own. In Sims’ *Quintet*, all combination tones are treated as consonances and other harmonics as dissonances.

Sims uses combination tones from the lowest two orders. *L’Homme Armé* uses combination tones from the lowest three orders, and treats harmonies with the simplest integer ratios (i.e. low prime factors) as more consonant than those with more complex ratios (i.e. higher prime factors). *Chorale:Angelswort* is not in Just Intonation and can not be accurately reproduced with any tuning system. Instead it migrates freely, maintaining only the combination-tone relationships between the lines. In a sense there
is only one consonance in this movement, the Golden Chord. This movement uses only Fibonacci combination tones, four specific tones from the Fibonacci-numbered orders.

In *Chorale:Angelswort* the relationships between all the voices in the chorale are maintained throughout the movement. In *L’Homme Armé*, only the generating tones are consistent throughout, while the other voices are freely assigned one of the six combination tones from the lowest three orders. In Sims’ *Quintet*, no particular relationship between the instruments is maintained. All are freely composed of harmonics that constitute any set of combination tones over the given fundamental.

The goal of harmonic motion in *L’Homme Armé* is tonal-sounding chords, while *Chorale:Angelswort* uses the Golden Chord as the harmonic focal point. Though the effect of the arrival is quite different, the process is related in that the harmonic motion has a logical goal based on the chosen combination-tone structure. In *L’Homme Armé*, the chosen combination tones are \([P], [Q], [P+Q], [P-Q], [2P+Q], [2P-Q], [2Q+P],\) and \([2Q-P]\). The variety of multiple sum and difference tones with a tonal center of A, as implied by the generating voices, allowed for a logical progression to an Authentic cadence in which the combination tones resulted in the simplest harmonic relationships between voices, as demonstrated in the cadence to 5:4:3:2:1 in Figure 18. In *Chorale:Angelswort*, the cadence to the Golden chord, seen in Figures 27-29, is equally logical given the progressive sum/difference tone approach. Since this method approaches the Golden Chord no matter what the generating interval (see Figure 25), the logical resolution is the arrival to the equally spaced Golden Chord in which all the pitches reinforce one another.
Ezra Sims’ *Quintet* and my *Angelswort* both use combination tones as a guiding harmonic principle. As in many aspects of the compositional process, the salient differences between these examples stem from the precise limitations the composer places upon him/herself. This evokes the oft-cited Stravinsky quotation: “My freedom thus consists in my moving about within the narrow frame that I have assigned myself for each one of my undertakings. I shall go even further: my freedom will be so much the greater and more meaningful, the more narrowly I limit my field of action and the more I surround myself with obstacles.”

Since 12-Tone Equal Temperament has come into question in the most recent half-century, many composers have sought new ways to create satisfying pitch relationships using microtones. From the simple Ring Modulation used in Stockhausen’s electroacoustic works of the 1950s and ’60’s, to Sims’ contrapuntal combination tones since the 1980s and more recently the inclusion of higher order limits in my *Angelswort*, combination tones continue to provide harmonic material for composers. The phenomenon is so innate to our perception of sound, it is not surprising that it has attracted composers as a method of organizing harmony. Combination tones are omnipresent in the acoustic world, and are reflected and amplified by composers.

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